

Energy flow in periodic thermodynamics

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A key quantity characterizing a time-periodically forced quantum system coupled to a heat bath is the energy flowing in the steady state through the system into the bath, where it is dissipated. We derive a general expression which allows one to compute this energy dissipation rate for a heat bath consisting of a large number of harmonic oscillators, and work out two analytically solvable model examples. In particular, we distinguish between genuine transitions effectuating a change of the system's Floquet state, and pseudo-transitions preserving that state; the latter are shown to yield an important contribution to the total dissipation rate. Our results suggest possible driving-mediated heating and cooling schemes on the quantum level. They also indicate that a driven system does not necessarily occupy only a single Floquet state when being in contact with a zero-temperature bath.

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I. INTRODUCTION

“Periodic thermodynamics”, a notion coined by W. Kohn, refers to the statistical physics of quantum systems which are driven by an arbitrarily strong time-periodic perturbation, and are weakly coupled to a heat bath [1]. Such systems have been considered in the context of, among others, Rydberg atoms driven by a monochromatic coherent microwave field in the presence of noise [2], open quantum systems under the influence of strong laser fields [3], driven dissipative quantum tunneling [4], and, more recently, degenerate Bose gases driven far from equilibrium [5]. In particular, Breuer *et al.* have emphasized the existence of a quasistationary distribution of Floquet-state occupation probabilities to which the system relaxes in the long-time limit under the combined effect of the time-periodic force and the heat bath [6]; this line of investigation has been taken up by Ketzmerick and Wustmann with a detailed view on the classical-quantum correspondence [7]. Even in this steady state, energy is continuously being fed by the driving force into the system and transported to the bath, where it is dissipated. This steady-state energy flow is one of the most important quantities characterizing a time-periodically driven open quantum system.

In the present paper we discuss the calculation of the energy dissipation rate from a conceptual point of view. We employ a golden rule-type perturbational approach which tends to gloss over certain theoretical details showing up in more elaborate treatments based on a Lindblad master equation [8–12], but which yields the same results when the Born-Markov approximation is made, and which has the merit of making the physical content of the central expression (38) for the energy dissipation rate particularly transparent. We proceed as follows: In Sec. II we discuss the golden rule for transitions among Floquet states. Although this is already implicitly contained in previous works [6, 10] we here give a detailed derivation, since this approach is capable of some generalizations. We then use this golden rule in Sec. III for de-

termining the energy dissipation rate for a time-periodically driven quantum system interacting with a thermal heat bath of harmonic oscillators. Here we distinguish between a contribution due to pseudo-transitions, which do not change the system's Floquet state, and the one due to genuine Floquet transitions. In the subsequent two sections we study two analytically solvable model systems, with emphasis placed on the connection between the dissipation rate and the ac Stark shift. In Sec. IV we briefly reconsider the linearly forced harmonic oscillator [6], which shows a fairly uncommon feature: All its quasienergy levels exhibit exactly the same ac Stark shift, so that the energy dissipation rate is entirely due to the pseudo-transitions. In contrast, the two-level system interacting with a circularly polarized radiation field investigated in Sec. V possesses a more generic level response, and a correspondingly richer dissipation pattern. Some conclusions are drawn in the final Sec. VI.

II. “GOLDEN RULE” FOR TRANSITIONS AMONG FLOQUET STATES

Consider a quantum system governed for times $t < 0$ by a Hamiltonian $H_0(t)$ which is periodic in time with period T ,

$$H_0(t) = H_0(t + T), \quad (1)$$

so that we have the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi^{(0)}(t)\rangle = H_0(t) |\psi^{(0)}(t)\rangle \quad (2)$$

for $t < 0$. We assume this system to possess a complete set of square-integrable Floquet states, that is, particular wave functions of the form [13–15]

$$|\psi_n^{(0)}(t)\rangle = |u_n(t)\rangle \exp(-i\varepsilon_n t/\hbar), \quad (3)$$

with real quasienergies ε_n and Floquet functions $|u_n(t)\rangle$ which inherit the periodicity of their Hamiltonian,

$$|u_n(t)\rangle = |u_n(t + T)\rangle. \quad (4)$$

The general solution to Eq. (2) then takes the form of a superposition of these states,

$$|\psi^{(0)}(t)\rangle = \sum_n c_n |u_n(t)\rangle \exp(-i\varepsilon_n t/\hbar), \quad (5)$$

with coefficients c_n which remain constant in time. This has the physically appealing consequence that one can assign constant occupation probabilities $|c_n|^2$ to the Floquet states, despite the Hamiltonian's explicit time-dependence. An additional perturbation acting on the system for $t > 0$ will then induce transitions among the Floquet states, so that one can ask how their occupation numbers change in response to that perturbation, *i.e.*, what the corresponding transition probabilities are; this is the question that will be tackled in the present section.

The proposition concerning the existence of such Floquet states (3) involves some mathematical subtleties which narrow down the range of systems to which the following deliberations can be applied rigorously, and which therefore deserve to be spelled out in some detail. These complications derive from a simple observation: Defining $\omega = 2\pi/T$, one has the obvious identity

$$\begin{aligned} & |u_n(t)\rangle \exp(-i\varepsilon_n t/\hbar) \\ &= |u_n(t)\rangle e^{ir\omega t} \exp(-i[\varepsilon_n + r\hbar\omega]t/\hbar), \end{aligned} \quad (6)$$

where the function $|u_n(t)\rangle e^{ir\omega t}$ again is periodic in time with period T , if r is any positive or negative integer. Thus, the separation of a Floquet state (3) into a periodic function $|u_n(t)\rangle$ and its Floquet multiplier $\exp(-i\varepsilon_n t/\hbar)$ is not unique: One is always free to choose any integer r , and then to replace $|u_n(t)\rangle$ by $|u_n(t)\rangle e^{ir\omega t}$, if one simultaneously replaces ε_n by $\varepsilon_n + r\hbar\omega$. That is, in contrast to the energy of an eigenstate of some time-independent Hamiltonian a quasienergy is not defined uniquely, but only up to an integer multiple of $\hbar\omega$. For instance, one could factorize the Floquet states such that all quasienergies fall into the “first Brillouin zone” $-\hbar\omega/2 \leq \varepsilon < +\hbar\omega/2$. It needs to be stressed, however, that this procedure would merely be a matter of convention and other choices are equally possible: As long as one considers the full Floquet states (3), rather than Floquet functions and quasienergies separately, the particular choice of the integer r involved in the formal factorization (6) is devoid of any significance.

Nonetheless, this Brillouin-zone structure of the quasienergy spectrum is the root of severe mathematical difficulties. Namely, assume that the Hamiltonian has the natural form $H_0(t) = K + \lambda W(t)$, where K defines an “unperturbed system” on which a time-periodic influence $W(t) = W(t+T)$ acts with adjustable strength λ . Then for $\lambda = 0$ the system's quasienergy spectrum consists of the energy eigenvalues of K , taken modulo $\hbar\omega$. Assuming further that K possesses infinitely many discrete energy eigenvalues, the corresponding quasienergy spectrum of $H_0(t)$ for $\lambda = 0$ generically covers the entire energy axis densely. The decisive question then is whether the quasienergy spectrum still remains a dense

pure point spectrum when $\lambda > 0$, or whether it becomes continuous: In the first case the sum expansion (5) is to be taken literally, so that the system's wave function is strictly quasiperiodic in time, whereas in the second case the continuous quasienergy spectrum gives rise to diffusive energy growth [16]. This question concerning the nature of the quasienergy spectrum of periodically time-dependent quantum systems is known as the “quantum stability problem” [17, 18]; since its solution heavily involves operator-theoretic versions of the Kolmogorov-Arnold-Moser theorem, it has drawn substantial interest in mathematical physics. Along this line of research, an important rigorous result is due to Howland: Provided the gap between successive energy eigenvalues of K grows sufficiently rapidly, and $W(t)$ is bounded, the quasienergy spectrum pertaining to $K + \lambda W(t)$ has no absolutely continuous component [19]; this finding has later been generalized by Joye [20]. Wishing to avoid unnecessary mathematical complications, but still aiming at physically meaningful statements, we restrict ourselves to such systems which do not admit an absolutely continuous quasienergy spectrum. The remaining class of systems still includes interesting and important models such as linearly forced anharmonic oscillators with superquadratic potentials [21], or, as a limiting case, the “driven particle in a box” [6].

We now stipulate that the system (2) be prepared in an individual Floquet state $n = i$ for $t < 0$, and then subjected to some perturbation $V(t)$ with an arbitrary time-dependence for $t > 0$, so that the evolution of the wave functions is given by

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = (H_0(t) + V(t)) |\psi(t)\rangle \quad (7)$$

for $t > 0$. In order to set up time-dependent perturbation theory within the Floquet framework, we adapt the standard textbook reasoning for evaluating transitions between energy eigenstates [22]: We introduce the time-evolution operator $U_0(t)$ of the unperturbed, periodically time-dependent system (2), which obeys the equation

$$i\hbar \dot{U}_0(t) = H_0(t) U_0(t) \quad (8)$$

with the initial condition $U_0(0) = 1$, and then employ this operator for transforming the wave functions $|\psi(t)\rangle$ to a Floquet-interaction picture by means of the relation

$$|\psi(t)\rangle = U_0(t) |\psi(t)\rangle_I. \quad (9)$$

The transformed wave function $|\psi(t)\rangle_I$ then evolves according to the equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle_I = V_I(t) |\psi(t)\rangle_I, \quad (10)$$

where

$$V_I(t) = U_0^\dagger(t) V(t) U_0(t) \quad (11)$$

denotes the perturbation operator transformed to the interaction picture, leading to the exact integral equation

$$|\psi(t)\rangle_I = |\psi(0)\rangle_I + \frac{1}{i\hbar} \int_0^t d\tau V_I(\tau) |\psi(\tau)\rangle_I. \quad (12)$$

It may be helpful to point out the rationale underlying this procedure: One might likewise convert the original evolution equation (7) into an integral equation, thus remaining fully within the Schrödinger picture; if one could solve that equation exactly the detour to the interaction picture were dispensable. However, the actual benefit of integral equations of the type (12) lies in the fact that they lend themselves to an iterative solution, leading to a Neumann series [23]. Then the interaction picture offers a tremendous advantage over the Schrödinger picture: Since the norm of the integral operator on the right hand side of Eq. (12) is determined by the assumedly small perturbation $V_I(t)$, the convergence properties of its formal series solution can be expected to be significantly better than those of the corresponding series obtained in the Schrödinger picture. Therefore, one may obtain acceptable approximations when terminating the interaction-picture Neumann series at low orders; in this sense the Floquet-interaction picture shares the virtues of the usual interaction picture employed in time-dependent perturbation theory [22].

For computing the probability of a transition from the initial Floquet state $n = i$ to some final Floquet state $n = f$ we require the projections

$$\langle u_f(t) | \psi(t) \rangle = \langle u_f(t) | U_0(t) | \psi(t) \rangle_I. \quad (13)$$

At this point, the fact that the unperturbed Hamiltonian (1) depends *periodically* on time becomes decisive: While in the case of an arbitrary time-dependence the evolution operator $U_0(t)$ would have to be expressed as a time-ordered exponential [22], here we have the Floquet representation

$$U_0(t) = \sum_n e^{-i\varepsilon_n t/\hbar} |u_n(t)\rangle \langle u_n(0)|, \quad (14)$$

giving

$$\langle u_f(t) | \psi(t) \rangle = e^{-i\varepsilon_f t/\hbar} \langle u_f(0) | \psi(t) \rangle_I. \quad (15)$$

To first order in V , the solution to the integral equation (12) now reads

$$|\psi(t)\rangle_I = |u_i(0)\rangle + \frac{1}{i\hbar} \int_0^t d\tau V_I(\tau) |u_i(0)\rangle. \quad (16)$$

Inserting the definition (11) and again utilizing the Floquet representation (14), one evaluates

$$\langle u_f(0) | V_I(\tau) | u_i(0) \rangle = e^{-i(\varepsilon_i - \varepsilon_f)\tau/\hbar} \langle u_f(\tau) | V(\tau) | u_i(\tau) \rangle, \quad (17)$$

from which we immediately obtain the desired transition probability P_{fi} for $f \neq i$ to lowest order in the perturbation V :

$$\begin{aligned} P_{fi} &= |\langle u_f(t) | \psi(t) \rangle|^2 \\ &= \frac{1}{\hbar^2} \left| \int_0^t d\tau e^{i(\varepsilon_f - \varepsilon_i)\tau/\hbar} \langle u_f(\tau) | V(\tau) | u_i(\tau) \rangle \right|^2. \end{aligned} \quad (18)$$

The apparent similarity of this result to the corresponding expression for transition probabilities among energy eigenstates [22] once again emphasizes the fact that in periodically time-dependent quantum systems the Floquet states take over the role which the energy eigenstates play in a system governed by a time-independent Hamiltonian.

To proceed, we assume that the perturbation is instantaneously switched on at time $t = 0$ and then stays constant,

$$V(t) = \begin{cases} 0 & ; \quad t < 0 \\ V & ; \quad t \geq 0. \end{cases} \quad (19)$$

Expanding the Floquet functions into Fourier series,

$$|u_n(t)\rangle = \sum_{k=-\infty}^{\infty} |u_n^{(k)}\rangle e^{ik\omega t}, \quad (20)$$

we obtain the Floquet transition matrix elements in the form

$$\begin{aligned} \langle u_f(\tau) | V | u_i(\tau) \rangle &= \sum_{k,j} \langle u_f^{(k)} | V | u_i^{(j)} \rangle e^{i(j-k)\omega\tau} \\ &= \sum_{\ell} e^{i\ell\omega\tau} V_{fi}^{(\ell)}, \end{aligned} \quad (21)$$

where

$$V_{fi}^{(\ell)} = \sum_k \langle u_f^{(k)} | V | u_i^{(k+\ell)} \rangle, \quad (22)$$

and Eq. (18) yields the transition probabilities

$$P_{fi} = \frac{1}{\hbar^2} \left| \sum_{\ell} \int_0^t d\tau e^{i(\varepsilon_f - \varepsilon_i + \ell\hbar\omega)\tau/\hbar} V_{fi}^{(\ell)} \right|^2. \quad (23)$$

When evaluating the squared sum, the cross-terms average to zero over a few cycles, so that one is left with

$$\begin{aligned} P_{fi} &\approx \frac{t^2}{\hbar^2} \sum_{\ell} \frac{\sin^2((\varepsilon_f - \varepsilon_i + \ell\hbar\omega)t/2\hbar)}{((\varepsilon_f - \varepsilon_i + \ell\hbar\omega)t/2\hbar)^2} |V_{fi}^{(\ell)}|^2 \\ &\sim \frac{2\pi}{\hbar} t \sum_{\ell} |V_{fi}^{(\ell)}|^2 \delta(\varepsilon_f - \varepsilon_i + \ell\hbar\omega) \end{aligned} \quad (24)$$

for intervals t which are, on the one hand, sufficiently long to allow for the replacement of the above squared sinc functions by delta distributions, but remain sufficiently

short to justify the first-order approximation (16) on the other.

This expression (24) constitutes the desired analog of the “golden rule” for transitions among Floquet states. Because each Floquet state brings its own set of Fourier components (20) into the dynamics, a transition $i \rightarrow f$ does not merely correspond to a single spectral line, but rather to a series of lines equally spaced by $\hbar\omega$, being weighted with the squared sum (22) of the components’ matrix elements.

III. ENERGY FLOW THROUGH DRIVEN QUANTUM SYSTEMS

To take the next step, we imagine that the system described by $H_0(t)$ is coupled to an environment conforming to a Hamiltonian H_B . With $H_0(t)$ acting on the system’s Hilbert space \mathcal{H}_S , and H_B acting on the space \mathcal{H}_B pertaining to the environmental degrees of freedom, the total Hamiltonian

$$H(t) = H_0(t) \otimes 1 + 1 \otimes H_B + H_{\text{int}} \quad (25)$$

then is defined on the product space $\mathcal{H}_S \otimes \mathcal{H}_B$. Here we take the coupling to be of the form

$$H_{\text{int}} = V \otimes W, \quad (26)$$

with V carrying the dimension of an energy, so that W is dimensionless. The previous reasoning leading to Eq. (24) can then easily be adapted: Assuming H_B to possess eigenstates $|\varphi_n\rangle$ with energies E_n , we have the replacements

$$\begin{aligned} |u_\alpha(t)\rangle &\rightarrow |u_\alpha(t)\rangle \otimes |\varphi_n\rangle \\ \varepsilon_\alpha &\rightarrow \varepsilon_\alpha + E_n, \end{aligned} \quad (27)$$

thus obtaining the rates

$$\Gamma_{fi}^{mn} = \frac{2\pi}{\hbar} \sum_\ell \left| V_{fi}^{(\ell)} \right|^2 |W_{mn}|^2 \delta(E_m - E_n + \hbar\omega_{fi}^\ell) \quad (28)$$

for individual system-environment transitions $(i, n) \rightarrow (f, m)$, where $W_{mn} = \langle \varphi_m | W | \varphi_n \rangle$, and

$$\omega_{fi}^\ell = (\varepsilon_f - \varepsilon_i)/\hbar + \ell\omega \quad (29)$$

denotes the frequencies associated with the system’s Floquet transition $i \rightarrow f$. Note that our approach neglects a second-order shift of the quasienergies which is induced by the interaction with the environment, but usually is quite small [2]. In the following we restrict ourselves to environments which can be described as a “heat bath” consisting of a very large number of thermally occupied harmonic oscillators [11]. Accordingly,

$$W = \sum_{\tilde{\omega}} \left(b_{\tilde{\omega}} + b_{\tilde{\omega}}^\dagger \right) \quad (30)$$

is a sum over all bath annihilation operators $b_{\tilde{\omega}}$ and their adjoint creation operators $b_{\tilde{\omega}}^\dagger$. We then have to distinguish two cases: If $E_n - E_m = \hbar\tilde{\omega} > 0$, so that the system gains the energy $\hbar\tilde{\omega}$ while the bath is de-excited and a bath phonon of frequency $\tilde{\omega}$ is annihilated, one has

$$W_{mn} = \sqrt{n(\tilde{\omega})}, \quad (31)$$

where $n(\tilde{\omega})$ is the initial occupation number of the bath oscillator involved in the transition. If, on the other hand, $E_n - E_m = \hbar\tilde{\omega} < 0$, so that the system loses energy to the bath and a phonon of frequency $\tilde{\omega}$ is created, one obtains

$$W_{mn} = \sqrt{n(\tilde{\omega}) + 1}. \quad (32)$$

Therefore, introducing the appropriate thermally averaged quantities [6]

$$N(\tilde{\omega}) = \begin{cases} \langle n(\tilde{\omega}) \rangle &= \frac{1}{e^{\beta\hbar\tilde{\omega}} - 1} & \text{when } \tilde{\omega} > 0 \\ \langle n(-\tilde{\omega}) \rangle + 1 &= \frac{e^{-\beta\hbar\tilde{\omega}}}{e^{-\beta\hbar\tilde{\omega}} - 1} & \text{when } \tilde{\omega} < 0, \end{cases} \quad (33)$$

where $\beta = 1/(k_B T)$ specifies the temperature T of the oscillator bath, and invoking its spectral density $J(\tilde{\omega})$, the total rate

$$\Gamma_{fi} = \int_{-\infty}^{+\infty} d\tilde{\omega} J(|\tilde{\omega}|) \frac{2\pi}{\hbar^2} \sum_\ell \left| V_{fi}^{(\ell)} \right|^2 N(\tilde{\omega}) \delta(-\tilde{\omega} + \omega_{fi}^\ell) \quad (34)$$

of bath-induced Floquet transitions $i \rightarrow f$ can be written as a sum,

$$\Gamma_{fi} = \sum_\ell \Gamma_{fi}^{(\ell)}, \quad (35)$$

with partial rates being given by

$$\Gamma_{fi}^{(\ell)} = \frac{2\pi}{\hbar^2} \left| V_{fi}^{(\ell)} \right|^2 N(\omega_{fi}^\ell) J(|\omega_{fi}^\ell|). \quad (36)$$

At this point, let us briefly check the consequences of factorizing the Floquet states $|u_n(t)\rangle \exp(-i\varepsilon_n t/\hbar)$ in different manners: If one replaces, for instance, $|u_f(t)\rangle$ by $|u_f(t)\rangle e^{ir\omega t}$, and ε_f by $\varepsilon_f + r\hbar\omega$ with arbitrary integer r , the matrix elements (22) are relabeled to read $V_{fi}^{(\ell-r)}$, while the transition frequencies (29) are referred to as $\omega_{fi}^{(\ell-r)}$, both retaining their numerical values. Thus, the net effect of this replacement is a relabeling of $\Gamma_{fi}^{(\ell)}$ to $\Gamma_{fi}^{(\ell-r)}$. Therefore, the choice of the representative of each Floquet function, that is, the choice of the respective integer r , merely is a matter of convenience.

Asking now for the steady-state distribution $\{p_n\}$ of Floquet-state occupation probabilities p_n which establishes itself under the influence of the bath, one has to solve the Pauli-type master equation [5–7]

$$\dot{p}_n = 0 = \sum_m (\Gamma_{nm} p_m - \Gamma_{mn} p_n) \quad (37)$$

into which only the total rates (35) enter. However, when asking for the rate R of energy dissipated in this steady state, a more detailed view is required: As emphasized in the above derivation, a partial rate $\Gamma_{fi}^{(\ell)}$ belongs to a transition over the course of which the system acquires the energy $\hbar\omega_{fi}^\ell$ from the bath. Therefore, the desired dissipation rate is determined by the sum over all these partial rates, each weighted with the occupation probability p_i of the respective initial state, and multiplied by the energy $-\hbar\omega_{fi}^\ell$ lost to the bath:

$$R = - \sum_{mnl} \hbar\omega_{mn}^\ell \Gamma_{mn}^{(\ell)} p_n . \quad (38)$$

It will be of interest for the subsequent discussion to observe that this dissipation rate (38) can naturally be decomposed into two parts: One contribution arising from genuine transitions $n \rightarrow m$ between *different* Floquet states n and m , and another one due to the “pseudo-transitions” $n \rightarrow n$. The former contribution is written as

$$R_{\text{trans}} = - \sum'_{mnl} \hbar\omega_{mn}^\ell \Gamma_{mn}^{(\ell)} p_n , \quad (39)$$

where the prime at the sum sign is meant to enforce the condition $m \neq n$, whereas the latter takes the form

$$R_{\text{pseudo}} = -\hbar\omega \sum_{n,\ell>0} \ell \left(\Gamma_{nn}^{(\ell)} - \Gamma_{nn}^{(-\ell)} \right) p_n, \quad (40)$$

having used $\omega_{nn}^\ell = \ell\omega$ in accordance with Eq. (29). Since the diagonal matrix element $\langle u_n(t) | V | u_n(t) \rangle$ is real, its Fourier component $V_{nn}^{(-\ell)}$ equals the complex conjugate of $V_{nn}^{(\ell)}$. Observing further that the definition (33) implies

$$N(\ell\omega) - N(-\ell\omega) = -1 \quad (41)$$

for $\ell > 0$, we find

$$R_{\text{pseudo}} = +\hbar\omega \sum_{n,\ell>0} \frac{2\pi}{\hbar^2} \ell \left| V_{nn}^{(\ell)} \right|^2 J(\ell\omega) p_n . \quad (42)$$

Thus, the pseudo-transitions yield a positive contribution to the dissipation rate, and make sure that energy is dissipated even in those cases in which the bath-induced genuine Floquet transitions do not figure.

IV. THE LINEARLY FORCED HARMONIC OSCILLATOR

An unusually simple, but still quite instructive model system is provided by a particle of mass M which is moving in a one-dimensional quadratic potential with oscillation frequency ω_0 while being subjected to a sinusoidal force with amplitude F and angular frequency $\omega \neq \omega_0$, as described in the position representation by the Hamiltonian

$$H(x, t) = -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial x^2} + \frac{1}{2} M \omega_0^2 x^2 + F x \cos(\omega t) . \quad (43)$$

The construction of its Floquet states follows a route laid out by Husimi [24, 25]: With $T = 2\pi/\omega$, let $\xi(t)$ be the T -periodic solution to the classical equation of motion

$$M\ddot{\xi} = -M\omega_0^2 \xi - F \cos(\omega t) , \quad (44)$$

namely,

$$\xi(t) = \frac{F}{M(\omega^2 - \omega_0^2)} \cos(\omega t) . \quad (45)$$

Then the solutions to the time-dependent Schrödinger equation are given by superpositions of the wave functions

$$\begin{aligned} \psi_n(x, t) &= \chi_n(x - \xi(t)) e^{-iE_n t/\hbar} \\ &\times \exp\left(\frac{i}{\hbar} \left[M\dot{\xi}(t)(x - \xi(t)) + \int_0^t d\tau L(\tau) \right]\right) , \end{aligned} \quad (46)$$

where $\chi_n(x)$ is an oscillator eigenfunction with energy $E_n = \hbar\omega(n + 1/2)$, and

$$L(t) = \frac{1}{2} M \dot{\xi}^2 - \frac{1}{2} M \omega_0^2 \xi^2 - F \xi \cos(\omega t) \quad (47)$$

denotes the classical Lagrangian of the system, evaluated along the T -periodic trajectory (45). The Floquet functions $u_n(x, t)$ and their quasienergies ε_n are then easily obtained by extracting the component increasing linearly with time from the phase of the solutions (46), giving [26]

$$\begin{aligned} u_n(x, t) &= \chi_n(x - \xi(t)) \exp\left(\frac{i}{\hbar} \left[M\dot{\xi}(t)(x - \xi(t)) \right. \right. \\ &\quad \left. \left. + \int_0^t d\tau L(\tau) - \frac{t}{T} \int_0^T d\tau L(\tau) \right]\right) \end{aligned} \quad (48)$$

and

$$\begin{aligned} \varepsilon_n &= E_n - \frac{1}{T} \int_0^T d\tau L(\tau) \\ &= \hbar\omega_0(n + 1/2) + \frac{F^2}{4M(\omega^2 - \omega_0^2)} . \end{aligned} \quad (49)$$

This latter result (49) expresses a peculiarity of the harmonic oscillator: All its states respond in the same manner to the external force, that is, all its energy levels exhibit precisely the same ac Stark shift proportional to the square of the driving amplitude.

Imposing now a dipole-type interaction of the form

$$V = \gamma x , \quad (50)$$

the fact that the Floquet functions (48) essentially are harmonic-oscillator eigenfunctions following the classical trajectory (45) without change of shape greatly facilitates the calculation of the required matrix elements [6]:

$$\begin{aligned} \langle u_m | x | u_n \rangle &= \langle u_m(t) | x - \xi(t) | u_n(t) \rangle + \delta_{mn} \xi(t) \\ &= \sqrt{\frac{\hbar}{2M\omega_0}} \left(\sqrt{n} \delta_{m,n-1} + \sqrt{n+1} \delta_{m,n+1} \right) \\ &\quad + \delta_{mn} \frac{F}{M(\omega^2 - \omega_0^2)} \cos(\omega t) . \end{aligned} \quad (51)$$

This expression provides the matrix elements $V_{fi}^{(\ell)}$, and therefore allows one to determine the partial rates (36). On the one hand, the only nonzero rates associated with genuine Floquet transitions are

$$\begin{aligned}\Gamma_{n-1,n}^{(0)} = \Gamma_{n-1,n} &= \frac{\pi\gamma^2 J(\omega_0)}{\hbar M \omega_0} \frac{n e^{\beta\hbar\omega_0}}{e^{\beta\hbar\omega_0} - 1} \\ \Gamma_{n+1,n}^{(0)} = \Gamma_{n+1,n} &= \frac{\pi\gamma^2 J(\omega_0)}{\hbar M \omega_0} \frac{n+1}{e^{\beta\hbar\omega_0} - 1}.\end{aligned}\quad (52)$$

On the other, each pseudo-transition $n \rightarrow n$ is characterized by the two partial rates

$$\begin{aligned}\Gamma_{nn}^{(1)} &= \frac{\pi\gamma^2 F^2 J(\omega)}{2\hbar^2 M^2 (\omega^2 - \omega_0^2)^2} \frac{1}{e^{\beta\hbar\omega} - 1} \\ \Gamma_{nn}^{(-1)} &= \frac{\pi\gamma^2 F^2 J(\omega)}{2\hbar^2 M^2 (\omega^2 - \omega_0^2)^2} \frac{e^{\beta\hbar\omega}}{e^{\beta\hbar\omega} - 1}.\end{aligned}\quad (53)$$

Now the master equation (37) determining the quasistationary Floquet distribution $\{p_n\}$ takes the form

$$\begin{aligned}0 = \dot{p}_n &= (\Gamma_{n,n-1} p_{n-1} - \Gamma_{n-1,n} p_n) \\ &+ (\Gamma_{n,n+1} p_{n+1} - \Gamma_{n+1,n} p_n).\end{aligned}\quad (54)$$

Adding the corresponding equation for \dot{p}_{n+1} effectuates an enhancement of the label n in the second bracket by one. Iterating this procedure, one deduces that the two brackets in this equation (54) have to vanish individually, giving

$$\frac{p_n}{p_{n-1}} = \frac{\Gamma_{n,n-1}}{\Gamma_{n-1,n}} = e^{-\beta\hbar\omega_0}.\quad (55)$$

Evidently, the uncommon feature that the total rates (35) consist, for this particular system (43), of only one partial rate ensures detailed balance of the Floquet transitions [6], so that the quasistationary Floquet occupation probabilities are given by a geometric Boltzmann distribution:

$$p_n = p_0 e^{-n\beta\hbar\omega_0}\quad (56)$$

for $n > 0$, while

$$p_0 = 1 - e^{-\beta\hbar\omega_0}.\quad (57)$$

Moreover, the fact that all quasienergies (49) differ from the unperturbed oscillator energies by the same ac Stark shift allows one to express this distribution as a Boltzmann distribution over these quasienergy levels [6]:

$$p_n = \frac{1}{Z} e^{-\beta\epsilon_n}\quad (58)$$

with the partition function

$$Z = \sum_{n=0}^{\infty} e^{-\beta\epsilon_n}.\quad (59)$$

Hence, the quasistationary Floquet distribution equals the canonical equilibrium distribution, regardless of the driving force.

Turning now to the energy dissipation rate in this steady state, the contribution (39) due to the genuine transitions here becomes

$$\begin{aligned}R_{\text{trans}} &= - \sum_n \left(\hbar\omega_{n+1,n}^{(0)} \Gamma_{n+1,n}^{(0)} p_n + \hbar\omega_{n-1,n}^{(0)} \Gamma_{n-1,n}^{(0)} p_n \right) \\ &= -\hbar\omega_0 \sum_n (\Gamma_{n,n-1} p_{n-1} - \Gamma_{n-1,n} p_n),\end{aligned}\quad (60)$$

which, in view of the detailed-balance condition (55), reduces to $R_{\text{trans}} = 0$. But there still remains the contribution (40) due to the pseudo-transitions:

$$R_{\text{pseudo}} = -\hbar\omega \sum_n \left(\Gamma_{nn}^{(1)} - \Gamma_{nn}^{(-1)} \right) p_n.\quad (61)$$

Since now Eqs. (53) yield

$$\Gamma_{nn}^{(1)} - \Gamma_{nn}^{(-1)} = -\frac{\pi\gamma^2 F^2 J(\omega)}{2\hbar^2 M^2 (\omega^2 - \omega_0^2)^2},\quad (62)$$

the sum over n corresponds to the normalization condition $\sum_n p_n = 1$, giving

$$R = R_{\text{pseudo}} = \hbar\omega \frac{\pi\gamma^2 F^2 J(\omega)}{2\hbar^2 M^2 (\omega^2 - \omega_0^2)^2}.\quad (63)$$

Thus, we have a fairly complete description of the energy flow through the driven harmonic oscillator (43): While the steady Floquet distribution (58) equals the thermal Boltzmann distribution over the unperturbed energy eigenstates for all parameters of the driving force, there is a continuous flow of energy through the system into the bath, which acts as an energy sink. This energy flow (63) is entirely due to pseudo-transitions which preserve the Floquet state, as expressed by the fact that the spectral density in Eq. (63) is to be evaluated at the driving frequency ω , not at the oscillator frequency ω_0 . The dissipation rate does not depend on the temperature of the bath, but grows quadratically with the driving amplitude, and becomes singular when the driving frequency approaches the oscillator frequency.

V. THE TWO-LEVEL SYSTEM IN A CIRCULARLY POLARIZED FIELD

A two-level system interacting with a circularly polarized monochromatic classical radiation field, as described by the Hamiltonian [27]

$$H_0(t) = \frac{1}{2} \hbar\omega_0 \sigma_z + \frac{\mu F}{2} (\sigma_x \cos \omega t + \sigma_y \sin \omega t),\quad (64)$$

defines a further analytically solvable model which, in spite of its quite minimalistic appearance, is already able

to reveal several generic features of the energy dissipation mechanism. Here σ_x , σ_y , and σ_z denote the usual Pauli matrices [22], and F quantifies the strength of the radiation field mode with frequency ω which couples to the bare two-level system with a constant μ . Thus, the energy eigenvalues of the unperturbed system are $E_{\pm} = \pm \hbar\omega_0/2$, so that ω_0 is the frequency of transitions between these bare levels. The formal simplicity of this model (64) is deceptive; in fact, its dynamics are by far richer than those of the driven harmonic oscillator (43).

A. Floquet functions and quasienergies

With the help of the Rabi frequency

$$\Omega = \sqrt{\delta^2 + (\mu F/\hbar)^2}, \quad (65)$$

where

$$\delta = \omega_0 - \omega \quad (66)$$

denotes the detuning of the driving frequency ω from the bare transition frequency ω_0 , the Floquet states of the driven two-level system (64) take the form [28]

$$|\psi_{\pm}(t)\rangle = e^{\mp i\Omega t/2} \frac{1}{\sqrt{2\Omega}} \begin{pmatrix} \pm\sqrt{\Omega \pm \delta} e^{-i\omega t/2} \\ \sqrt{\Omega \mp \delta} e^{+i\omega t/2} \end{pmatrix}. \quad (67)$$

From these states we split off the Floquet functions

$$|u_{\pm}(t)\rangle = \frac{1}{\sqrt{2\Omega}} \begin{pmatrix} \pm\sqrt{\Omega \pm \delta} \\ \sqrt{\Omega \mp \delta} e^{+i\omega t} \end{pmatrix}, \quad (68)$$

implying that the corresponding quasienergies read

$$\varepsilon_{\pm} = \frac{\hbar}{2}(\omega \pm \Omega). \quad (69)$$

Now an important distinction has to be made: If $\delta > 0$, so that the driving frequency is detuned to the red side of the bare transition, these quasienergies reduce to

$$\begin{aligned} \varepsilon_+ &\rightarrow +\hbar\omega_0/2 \\ \varepsilon_- &\rightarrow -\hbar\omega_0/2 + \hbar\omega \end{aligned} \quad (70)$$

in the limit $F \rightarrow 0$ of vanishing driving amplitude. In contrast, when the radiation field is blue-detuned and $\delta < 0$, these limits are given by

$$\begin{aligned} \varepsilon_+ &\rightarrow -\hbar\omega_0/2 + \hbar\omega \\ \varepsilon_- &\rightarrow +\hbar\omega_0/2. \end{aligned} \quad (71)$$

Thus, the Floquet state labeled by “+” exhibits a quasienergy which *increases* monotonically with increasing amplitude F ; this state is continuously connected to the *excited* state of the bare two-level system when $\delta > 0$, and to its *ground* state when $\delta < 0$; vice versa for the Floquet state labeled by “−”. Expressed differently, the ac Stark shift of the two levels changes *qualitatively* when δ changes its sign: As indicated in Fig. 1, the two levels repel each other with increasing amplitude in the case of red detuning, whereas they approach each other and cross for blue detuning. This characteristic behavior also leaves its traces in the energy dissipation rate.

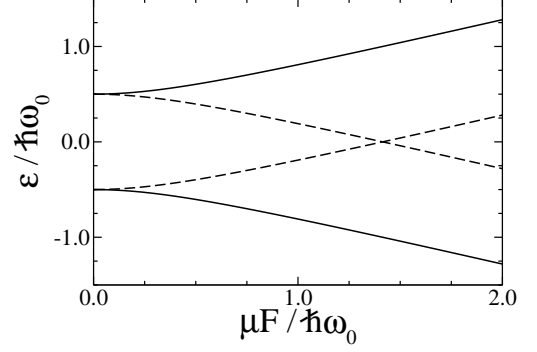


FIG. 1: Ac Stark shift for the two-level system driven by a circularly polarized radiation field: Shown are those representatives of the quasienergies ε which connect continuously to the bare energy levels $\pm \hbar\omega_0/2$. In the case of red detuning, when $\omega < \omega_0$, the two quasienergies repel each other with increasing scaled amplitude $\mu F/(\hbar\omega_0)$ (full lines: $\omega/\omega_0 = 0.5$), whereas they approach each other and cross for blue detuning, when $\omega > \omega_0$ (dashed lines: $\omega/\omega_0 = 1.5$).

B. Steady Floquet distribution

For modeling the system’s coupling to the bath, we choose

$$V = \gamma \sigma_x; \quad (72)$$

for simplicity, here we take the spectral density J of the bath to be constant. The Floquet functions (68) readily yield the matrix elements

$$\langle u_+ | V | u_- \rangle = \frac{\gamma}{2\Omega} ((\Omega + \delta)e^{i\omega t} - (\Omega - \delta)e^{-i\omega t}), \quad (73)$$

possessing only two nonvanishing Fourier components

$$V_{+-}^{(\pm 1)} = \pm \gamma \frac{\Omega \pm \delta}{2\Omega}; \quad (74)$$

likewise, one finds

$$V_{-+}^{(\pm 1)} = \mp \gamma \frac{\Omega \mp \delta}{2\Omega}. \quad (75)$$

For evaluating the partial rates (36) we need to know the sign of the associated transition frequencies ω_{fi}^ℓ , in order to resolve the distinction made in the definition (33) of $N(\omega_{fi}^\ell)$. Assuming $\Omega \leq \omega$, Eq. (69) leads to

$$\begin{aligned} \omega_{+-}^1 &= \Omega + \omega > 0 \\ \omega_{+-}^{-1} &= \Omega - \omega \leq 0 \\ \omega_{-+}^1 &= -\Omega + \omega \geq 0 \\ \omega_{-+}^{-1} &= -\Omega - \omega < 0; \end{aligned} \quad (76)$$

after introducing the convenient abbreviation

$$\Gamma_0 = \frac{2\pi\gamma^2 J}{\hbar^2} \quad (77)$$

one then finds the corresponding partial rates for the genuine Floquet transitions,

$$\begin{aligned}\Gamma_{+-}^{(1)} &= \frac{(\Omega + \delta)^2}{4\Omega^2} \frac{\Gamma_0}{e^{\beta\hbar(\omega+\Omega)} - 1} \\ \Gamma_{+-}^{(-1)} &= \frac{(\Omega - \delta)^2}{4\Omega^2} \frac{\Gamma_0 e^{\beta\hbar(\omega-\Omega)}}{e^{\beta\hbar(\omega-\Omega)} - 1} \\ \Gamma_{-+}^{(1)} &= \frac{(\Omega - \delta)^2}{4\Omega^2} \frac{\Gamma_0}{e^{\beta\hbar(\omega-\Omega)} - 1} \\ \Gamma_{-+}^{(-1)} &= \frac{(\Omega + \delta)^2}{4\Omega^2} \frac{\Gamma_0 e^{\beta\hbar(\omega+\Omega)}}{e^{\beta\hbar(\omega+\Omega)} - 1}.\end{aligned}\quad (78)$$

However, when $\Omega > \omega$, the signs of ω_{+-}^{-1} and ω_{-+}^1 are reversed, resulting in

$$\begin{aligned}\Gamma_{+-}^{(-1)} &= \frac{(\Omega - \delta)^2}{4\Omega^2} \frac{\Gamma_0}{e^{\beta\hbar(\Omega-\omega)} - 1} \\ \Gamma_{-+}^{(1)} &= \frac{(\Omega - \delta)^2}{4\Omega^2} \frac{\Gamma_0 e^{\beta\hbar(\Omega-\omega)}}{e^{\beta\hbar(\Omega-\omega)} - 1}.\end{aligned}\quad (79)$$

This information suffices to determine the quasistationary Floquet distribution $\{p_+, p_-\}$ for both cases: Starting from the master equation

$$0 = \Gamma_{+-}p_- - \Gamma_{-+}p_+ \quad (80)$$

and inserting $p_+ = 1 - p_-$, one has

$$\begin{aligned}p_- &= \frac{\Gamma_{-+}}{\Gamma_{-+} + \Gamma_{+-}} \\ &= \frac{\Gamma_{-+}^{(1)} + \Gamma_{-+}^{(-1)}}{\Gamma_{-+}^{(1)} + \Gamma_{-+}^{(-1)} + \Gamma_{+-}^{(1)} + \Gamma_{+-}^{(-1)}}.\end{aligned}\quad (81)$$

After some elementary calculation, this leads for $\Omega \leq \omega$ to

$$p_- = \frac{1}{2} + \frac{\Omega\delta [\cosh(\beta\hbar\omega) - \cosh(\beta\hbar\Omega)]}{(\Omega^2 + \delta^2) \sinh(\beta\hbar\omega) - 2\Omega\delta \sinh(\beta\hbar\Omega)}, \quad (82)$$

whereas for $\Omega > \omega$ we find

$$p_- = \frac{1}{2} + \frac{\frac{1}{2}(\Omega^2 + \delta^2) [\cosh(\beta\hbar\Omega) - \cosh(\beta\hbar\omega)]}{(\Omega^2 + \delta^2) \sinh(\beta\hbar\Omega) - 2\Omega\delta \sinh(\beta\hbar\omega)}. \quad (83)$$

Observe that $\Omega > \omega$ when $(\mu F/\hbar)^2 > 2\omega\omega_0 - \omega_0^2$, so that one always ends up in the regime $\Omega > \omega$ when the scaled driving amplitude $\mu F/(\hbar\omega_0)$ becomes sufficiently large. Eq. (83) then implies $p_- \rightarrow 1$ for $\mu F/(\hbar\omega_0) \rightarrow \infty$: For any finite temperature of the bath, the Floquet state with a “downward” ac Stark shift will acquire all the population in the strong-forcing limit, regardless of whether this state is connected to the ground state or to the excited state of the bare two-level system in the opposite limit of vanishing driving amplitude. Figure 2 depicts p_- vs. $\mu F/(\hbar\omega_0)$ for $\omega/\omega_0 = 1.5$, as corresponding to the ac Stark shift indicated by the dashed lines in Fig. 1. For low scaled temperature $k_B T/(\hbar\omega_0) = (\beta\hbar\omega_0)^{-1}$ the state labeled “−”, here being connected to the excited

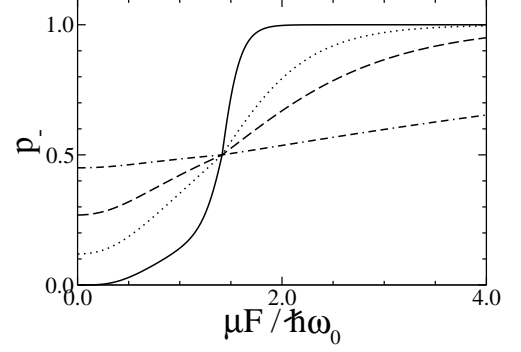


FIG. 2: Population of the Floquet state labeled “−” according to Eqs. (82) and (83) for $\omega/\omega_0 = 1.5$ and scaled temperatures $k_B T/(\hbar\omega_0) = 0.1$ (full line), 0.5 (dotted), 1.0 (dashed), and 5.0 (dash-dash-dotted).

state of the bare system, naturally is almost unpopulated when $\mu F/(\hbar\omega_0) \ll 1$, but it accepts practically the entire population when $\mu F/(\hbar\omega_0) \gtrsim 2$. For higher temperatures the “−”-state carries more population already in the weak-driving regime, and the crossover to the limit $p_- = 1$ is less pronounced, but this limit is approached with arbitrarily small deviation when $\mu F/(\hbar\omega_0)$ becomes large enough. If it were feasible to decouple the system from the heat bath at will, this would open up interesting heating and cooling schemes: Suppose than an ensemble of two-level systems is strongly driven, and that the ensemble’s contact with the heat bath is disabled when $p_- \approx 1$. If one then switches off the driving amplitude smoothly, the adiabatic principle for Floquet states [26] guarantees that the value of p_- remains practically unchanged. Therefore, if $\omega > \omega_0$ one obtains an “ultrahot” ensemble of bare two-level systems, with more or less all of its members being in the excited energy eigenstate at the end of the switch-off, whereas the final state would be an “ultracold” ensemble of two-level systems in their ground states when $\omega < \omega_0$.

Another remarkable feature revealed by the steady-state occupation probabilities (82) and (83) is their low-temperature limit in the presence of the drive: When $\Omega > \omega$, Eq. (83) simply yields $p_- \rightarrow 1$ for $\beta\hbar\omega_0 \rightarrow \infty$; in this case the “−”-state carries all the population for vanishing bath temperature. In contrast, when $\Omega < \omega$ and Eq. (82) applies, one has

$$p_- \rightarrow \frac{1}{2} + \frac{\Omega\delta}{\Omega^2 + \delta^2} \quad \text{for } \beta\hbar\omega_0 \rightarrow \infty. \quad (84)$$

If one now additionally takes the limit of vanishing driving amplitude, this expression (84) properly reduces to

$$p_- \rightarrow \frac{1}{2} + \frac{1}{2} \text{sign}(\delta) \quad \text{for } \mu F/(\hbar\omega_0) \rightarrow 0, \quad (85)$$

so that $p_- \rightarrow 1$ when $\delta > 0$ and the “−”-state becomes the bare ground state, whereas $p_- \rightarrow 0$ when $\delta < 0$ and

the “−”-state connects to the excited energy eigenstate of the bare two-level system. However, for finite nonzero driving strength matching the condition $\Omega < \omega$, none of the two Floquet states can accept all the population at zero temperature.

The physics behind this finding becomes clear if one takes the limit of vanishing temperature already at the level of the partial rates (78) and (79): For both $\Omega < \omega$ and $\Omega > \omega$ Eq. (78) gives, for $\beta\hbar\omega_0 \rightarrow \infty$,

$$\Gamma_{+-}^{(1)} \rightarrow 0 \quad , \quad \Gamma_{-+}^{(-1)} \rightarrow \frac{(\Omega + \delta)^2}{4\Omega^2} \Gamma_0 . \quad (86)$$

In addition, when $\Omega > \omega$ one obtains from Eq. (79)

$$\Gamma_{+-}^{(-1)} \rightarrow 0 \quad , \quad \Gamma_{-+}^{(1)} \rightarrow \frac{(\Omega - \delta)^2}{4\Omega^2} \Gamma_0 . \quad (87)$$

Hence, $\Gamma_{+-} = \Gamma_{+-}^{(1)} + \Gamma_{+-}^{(-1)}$ goes to zero in this case $\Omega > \omega$, while Γ_{-+} remains nonzero. Therefore, at very low temperatures the driven system can still undergo transitions from “+” to “−”, but not back from “−” to “+”, so that eventually all the population piles up in the “−”-state. In contrast, when $\Omega < \omega$ one deduces

$$\Gamma_{+-}^{(-1)} \rightarrow \frac{(\Omega - \delta)^2}{4\Omega^2} \Gamma_0 \quad , \quad \Gamma_{-+}^{(1)} \rightarrow 0 \quad (88)$$

from Eq. (78), so that now *both* $\Gamma_{+-} = \Gamma_{+-}^{(-1)}$ and $\Gamma_{-+} = \Gamma_{-+}^{(1)}$ remain nonzero, and transitions in both directions remain enabled even at vanishing temperature; using Eq. (81) one easily recovers Eq. (84) from the above rates. The fact that even at vanishing bath temperature both “upward” and “downward” transitions can remain active is a distinctive feature of periodic thermodynamics.

C. Energy dissipation rate

Collecting all nonvanishing contributions, the energy dissipation rate (38) for the circularly forced two-level system (64) becomes

$$\begin{aligned} R = & \hbar\omega \left(\Gamma_{++}^{(-1)} - \Gamma_{++}^{(1)} \right) (1 - p_-) \\ & + \hbar\omega \left(\Gamma_{--}^{(-1)} - \Gamma_{--}^{(1)} \right) p_- \\ & - \hbar(\Omega + \omega) \Gamma_{+-}^{(1)} p_- - \hbar(\Omega - \omega) \Gamma_{+-}^{(-1)} p_- \\ & - \hbar(-\Omega + \omega) \Gamma_{-+}^{(1)} (1 - p_-) \\ & - \hbar(-\Omega - \omega) \Gamma_{-+}^{(-1)} (1 - p_-) , \end{aligned} \quad (89)$$

where the first two terms on the right-hand side stem from the pseudo-transitions. Computing the associated partial rates

$$\begin{aligned} \Gamma_{\pm\pm}^{(1)} &= \frac{\Omega^2 - \delta^2}{4\Omega^2} \frac{\Gamma_0}{e^{\beta\hbar\omega} - 1} \\ \Gamma_{\pm\pm}^{(-1)} &= \frac{\Omega^2 - \delta^2}{4\Omega^2} \frac{\Gamma_0 e^{\beta\hbar\omega}}{e^{\beta\hbar\omega} - 1} \end{aligned} \quad (90)$$

and observing

$$\Gamma_{\pm\pm}^{(-1)} - \Gamma_{\pm\pm}^{(1)} = \Gamma_0 \frac{\Omega^2 - \delta^2}{4\Omega^2} , \quad (91)$$

we obtain

$$R_{\text{pseudo}} = \frac{\hbar\omega\Gamma_0}{4} \left(\frac{\mu F}{\hbar\Omega} \right)^2 , \quad (92)$$

having used the definition (65) of Ω . Next, the contribution of the genuine Floquet transitions is rearranged to read

$$\begin{aligned} R_{\text{trans}} = & \hbar\Omega \left(\Gamma_{-+}^{(-1)} + \Gamma_{-+}^{(1)} \right) + \hbar\omega \left(\Gamma_{-+}^{(-1)} - \Gamma_{-+}^{(1)} \right) \\ & - \hbar\Omega \left(\Gamma_{-+}^{(-1)} + \Gamma_{-+}^{(1)} + \Gamma_{+-}^{(1)} + \Gamma_{+-}^{(-1)} \right) p_- \\ & - \hbar\omega \left(\Gamma_{-+}^{(-1)} - \Gamma_{-+}^{(1)} + \Gamma_{+-}^{(1)} - \Gamma_{+-}^{(-1)} \right) p_- . \end{aligned} \quad (93)$$

Now the equation (81) for p_- effectuates the cancellation of the terms proportional to $\hbar\Omega$, and after some algebra one arrives at

$$R_{\text{trans}} = \frac{\hbar\omega\Gamma_0}{4} \frac{(\Omega^2 - \delta^2)^2}{\Delta^2\Omega^2} \sinh(\beta\hbar\omega) \quad (94)$$

with

$$\Delta^2 = (\Omega^2 + \delta^2) \sinh(\beta\hbar\Omega_>) - 2\Omega\delta \sinh(\beta\hbar\Omega_<) , \quad (95)$$

where $\Omega_>$ ($\Omega_<$) is the larger (smaller) of the two frequencies Ω and ω ; recall that this quantity Δ^2 also appears in the denominator of the steady-state occupation probabilities (82) and (83). In contrast to R_{pseudo} , this dissipation rate (94) caused by the genuine Floquet transitions does depend on the temperature of the bath. Adding the two contributions, we finally obtain the total dissipation rate

$$R = \frac{\hbar\omega\Gamma_0}{4} \left(\frac{\mu F}{\hbar\Omega} \right)^2 \left[1 + \left(\frac{\mu F}{\hbar\Delta} \right)^2 \sinh(\beta\hbar\omega) \right] . \quad (96)$$

Considering finite nonzero temperatures, this total dissipation rate evidently approaches a finite value determined solely by the pseudo-transitions in the strong-forcing regime,

$$R \rightarrow \frac{\hbar\omega\Gamma_0}{4} \quad \text{for } \mu F/(\hbar\omega_0) \rightarrow \infty , \quad (97)$$

whereas it vanishes in the high-frequency limit,

$$R \rightarrow 0 \quad \text{for } \omega/\omega_0 \rightarrow \infty . \quad (98)$$

In Fig. 3 we plot the normalized (dimensionless) rate $R^{(0)} = R/(\hbar\omega_0\Gamma_0)$ for precisely the cases studied previously in Fig. 2, that is, for $\omega/\omega_0 = 1.5$ and various scaled temperatures; here the approach to the strong-forcing limit (97) is already recognizable for $\mu F/(\hbar\omega_0) \approx 4$. A better understanding of the non-monotonic behavior of these curves is obtained if one investigates the dependence of the dissipation rate on the driving frequency at

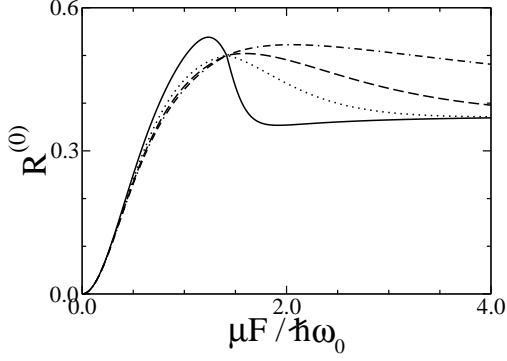


FIG. 3: Normalized energy dissipation rate $R^{(0)} = R/(\hbar\omega_0\Gamma_0)$ for the situations considered in Fig. 2, that is, for $\omega/\omega_0 = 1.5$ and scaled temperatures $k_B T/(\hbar\omega_0) = 0.1$ (full line), 0.5 (dotted), 1.0 (dashed), and 5.0 (dash-dash-dotted).

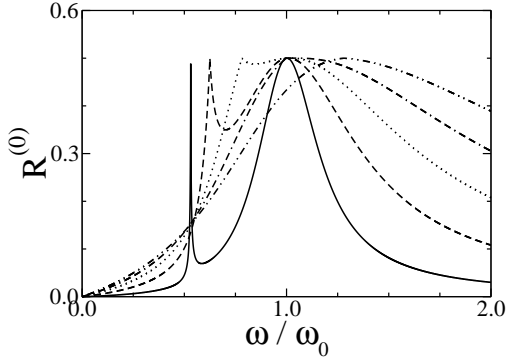


FIG. 4: Normalized energy dissipation rate $R^{(0)} = R/(\hbar\omega_0\Gamma_0)$ for the scaled temperature $k_B T/(\hbar\omega_0) = 1.0$, and scaled driving strengths $\mu F/(\hbar\omega_0) = 0.25$ (full line), 0.5 (dashed), 0.75 (dotted), 1.0 (dash-dash-dotted), and 1.25 (dash-dot-dotted).

fixed driving amplitude, as illustrated in Fig. 4: For low driving amplitudes conforming to $\mu F/(\hbar\omega_0) < 1$ one has a broad resonance at about $\omega = \omega_0$ and a further, narrow resonance at $\omega = \Omega$, which condition is equivalent to

$$\frac{\omega}{\omega_0} = \frac{1}{2} \left(1 + \left(\frac{\mu F}{\hbar\omega_0} \right)^2 \right). \quad (99)$$

Both resonances merge when $\mu F/(\hbar\omega_0) = 1$; for still higher driving amplitudes there is only one single, broad maximum of the dissipation rate at a position determined mainly by the pseudo-transitions, located at $\omega/\omega_0 \approx \sqrt{1 + (\mu F/\hbar\omega_0)^2}$ in the strong-forcing regime.

Once again, the limiting case of vanishing bath temperature merits special attention: When $\Omega > \omega$ one deduces

$$R_{\text{trans}} \rightarrow 0 \quad \text{for } \beta\hbar\omega_0 \rightarrow \infty \quad (100)$$

from Eq. (94), but when $\Omega < \omega$ we find

$$R_{\text{trans}} \rightarrow \frac{\hbar\omega\Gamma_0}{4} \left(\frac{\mu F}{\hbar\Omega} \right)^4 \frac{1}{1 + (\delta/\Omega)^2} \quad \text{for } \beta\hbar\omega_0 \rightarrow \infty. \quad (101)$$

Thus, in the zero-temperature limit the genuine Floquet transitions do not figure when $\Omega > \omega$, but they do yield a finite contribution to the total dissipation rate when $\Omega < \omega$. This distinction evidently matches the behavior of the steady-state occupation probabilities discussed at the end of the previous subsection.

VI. CONCLUSIONS

The concept of “periodic thermodynamics” [1] implies that the steady state to which a quantum system relaxes in the presence of both a time-periodic driving force and a heat bath continuously delivers energy to the bath. The calculation of this steady-state energy dissipation rate, which we have outlined here for an harmonic-oscillator bath, reveals some peculiar features: Whereas the steady-state occupation probabilities are determined by a Pauli-type master equation into which only the rates for the genuine Floquet transitions enter [6], the dissipation rate (38) also incorporates the contribution (42) from processes during which the bath energy changes by an integer multiple of $\hbar\omega$, while the system’s Floquet state is left unchanged; these processes have been dubbed pseudo-transitions. Moreover, only the total rates (35) embodying *all* Floquet transition frequencies show up in the master equation, whereas the evaluation of the dissipation rate (38) requires the knowledge of the individual partial rates (36).

The example of the driven harmonic oscillator considered in Sec. IV is quite instructive insofar as it shows what does *not* happen in generic cases: For this particular model the total transition rates (52) for the genuine transitions consist of only one partial rate, which implies that detailed balance still holds and the steady-state Floquet distribution equals the thermal Boltzmann distribution [6]; in addition, here the dissipation rate is given entirely by the pseudo-transitions. The study of the circularly forced two-level system performed in Sec. V has demonstrated that the steady-state dynamics are substantially more involved when the bare energy levels of the driven system exhibit a nontrivial ac Stark shift. The features encountered here will also show up, multiply superimposed, in non-integrable systems which require numerical treatment [5–7]. Particularly noteworthy is the fact that even in contact with a zero-temperature bath a time-periodically driven system does not necessarily occupy only one single Floquet state, as exemplified by the two-level model in the regime $\Omega < \omega$.

With a view towards future applications, the observations made at the end of Subsec. VB might merit further investigations and generalizations. The steady-state Floquet distribution which establishes itself in the presence

of the driving force may be preserved by decoupling the bath and switching off the driving amplitude adiabatically; the resulting state then can contain either more or even less energy than a stationary thermal state. Thus, as a matter of principle it seems possible to achieve cooling by driving.

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